

# A characterization of dissimilarity families of trees

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## Abstract

Let  $\mathcal{T} = (T, w)$  be a weighted finite tree with leaves  $1, \dots, n$ . For any  $I := \{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ , let  $D_I(\mathcal{T})$  be the weight of the minimal subtree of  $T$  connecting  $i_1, \dots, i_k$ ; the  $D_I(\mathcal{T})$  are called  $k$ -weights of  $\mathcal{T}$ . Given a family of real numbers parametrized by the  $k$ -subsets of  $\{1, \dots, n\}$ ,  $\{D_I\}_{I \in \binom{\{1, \dots, n\}}{k}}$ , we say that a weighted tree  $\mathcal{T} = (T, w)$  with leaves  $1, \dots, n$  realizes the family if  $D_I(\mathcal{T}) = D_I$  for any  $I$ .

In 2006 Levy, Yoshida and Pachter defined, for any positive-weighted tree  $\mathcal{T} = (T, w)$  with  $\{1, \dots, n\}$  as leaf set and any  $i, j \in \{1, \dots, n\}$ , the numbers  $S_{i,j}$  to be  $\sum_{Y \in \binom{\{1, \dots, n\} - \{i, j\}}{k-2}} D_{i,j,Y}(\mathcal{T})$ ; they proved that there exists a positive-weighted tree  $\mathcal{T}' = (T', w')$  such that  $D_{i,j}(\mathcal{T}') = S_{i,j}$  for any  $i, j \in \{1, \dots, n\}$  and that this new tree is, in some way, similar to the given one. In this paper, by using the  $S_{i,j}$  defined by Levy, Yoshida and Pachter, we characterize families of real numbers parametrized by  $\binom{\{1, \dots, n\}}{k}$  that are the families of  $k$ -weights of weighted trees with leaf set equal to  $\{1, \dots, n\}$  and weights of the internal edges positive.

## 1 Introduction

For any graph  $G$ , let  $E(G)$ ,  $V(G)$  and  $L(G)$  be respectively the set of the edges, the set of the vertices and the set of the leaves of  $G$ . A **weighted graph**  $\mathcal{G} = (G, w)$  is a graph  $G$  endowed with a function  $w : E(G) \rightarrow \mathbb{R}$ . For any edge  $e$ , the real number  $w(e)$  is called the weight of the edge. If all the weights are nonnegative (respectively positive), we say that the graph is **nonnegative-weighted** (respectively **positive-weighted**); if the weights of the internal edges are nonzero, we say that the graph is **internal-nonzero-weighted** and, if the weights of the internal edges are positive, we say that the graph is **internal-positive-weighted**. For any finite subgraph  $G'$  of  $G$ , we define  $w(G')$  to be the sum of the weights of the edges of  $G'$ . In this paper we will deal only with weighted finite trees.

**Definition 1.** Let  $\mathcal{T} = (T, w)$  be a weighted tree. For any distinct  $i_1, \dots, i_k \in V(T)$ , we define  $D_{\{i_1, \dots, i_k\}}(\mathcal{T})$  to be the weight of the minimal subtree containing  $i_1, \dots, i_k$ . We call this subtree “the subtree realizing  $D_{\{i_1, \dots, i_k\}}(\mathcal{T})$ ”. More simply, we denote  $D_{\{i_1, \dots, i_k\}}(\mathcal{T})$  by  $D_{i_1, \dots, i_k}(\mathcal{T})$  for any order of  $i_1, \dots, i_k$ . We call the  $D_{i_1, \dots, i_k}(\mathcal{T})$  the  $k$ -**weights** of  $\mathcal{T}$  and we call a  $k$ -weight of  $\mathcal{T}$  for some  $k$  a **multiweight** of  $\mathcal{T}$ .

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If  $S$  is a subset of  $V(T)$ , the  $k$ -weights  $D_{i_1, \dots, i_k}(\mathcal{T})$  with  $i_1, \dots, i_k \in S$  give a vector in  $\mathbb{R}^{\binom{S}{k}}$ . This vector is called  **$k$ -dissimilarity vector** of  $(\mathcal{T}, S)$ . Equivalently, we can speak of the **family of the  $k$ -weights** of  $(\mathcal{T}, S)$  or of the  **$k$ -dissimilarity family of  $(\mathcal{T}, S)$** .

If  $S$  is a finite set,  $k \in \mathbb{N}$  and  $k < \#S$ , we say that a family of real numbers  $\{D_I\}_{I \in \binom{S}{k}}$  is **treelike** (respectively p-treelike, nn-treelike, inz-treelike, ip-treelike) if there exist a weighted (respectively positive-weighted, nonnegative-weighted, internal-nonzero-weighted, internal-positive-weighted) tree  $\mathcal{T} = (T, w)$  and a subset  $S$  of the set of its vertices such that  $D_I(\mathcal{T}) = D_I$  for any  $k$ -subset  $I$  of  $S$ . In this case, we say also that  $\mathcal{T}$  realizes the family  $\{D_I\}_{I \in \binom{S}{k}}$ . If in addition  $S \subset L(T)$ , we say that the family is **l-treelike** (respectively p-l-treelike, nn-l-treelike, inz-l-treelike, ip-l-treelike).

Weighted graphs have applications in several disciplines, such as biology and psychology. Phylogenetic trees are weighted graphs whose vertices represent species and the weight of an edge is given by how much the DNA sequences of the species represented by the vertices of the edge differ. Dissimilarity families arise naturally also in psychology, see for instance the introduction in [7]. There is a wide literature concerning graphlike dissimilarity families and treelike dissimilarity families, in particular concerning methods to reconstruct weighted trees from their dissimilarity families; these methods, for instance the so-called neighbor-joining method, are used by biologists to reconstruct phylogenetic trees. See for example [13], [19] and [8], [17] for overviews. We recall the most important results concerning treelike dissimilarity families.

A criterion for a metric on a finite set to be nn-l-treelike was established in [6], [18], [20]:

**Definition 2.** Let  $\{D_I\}_{I \in \binom{\{1, \dots, n\}}{2}}$  be a family of positive real numbers. We say that the  $D_I$  satisfy the **4-point condition** if and only if for all distinct  $a, b, c, d \in \{1, \dots, n\}$ , the maximum of

$$\{D_{a,b} + D_{c,d}, D_{a,c} + D_{b,d}, D_{a,d} + D_{b,c}\}$$

is attained at least twice.

**Theorem 3.** Let  $\{D_I\}_{I \in \binom{\{1, \dots, n\}}{2}}$  be a family of positive real numbers satisfying the triangle inequalities. It is p-treelike (or nn-l-treelike) if and only if the 4-point condition holds.

Also the study of general weighted trees can be interesting and, in [3], Bandelt and Steel proved a result, analogous to Theorem 3, for general weighted trees:

**Theorem 4. (Bandelt-Steel)** For any family of real numbers  $\{D_I\}_{I \in \binom{\{1, \dots, n\}}{2}}$ , there exists a weighted tree  $\mathcal{T}$  with leaves  $1, \dots, n$  such that  $D_I(\mathcal{T}) = D_I$  for any  $I \in \binom{\{1, \dots, n\}}{2}$  if and only if the so-called relaxed 4-point condition holds, i.e. for any  $a, b, c, d \in \{1, \dots, n\}$ , at least two among  $D_{a,b} + D_{c,d}$ ,  $D_{a,c} + D_{b,d}$ ,  $D_{a,d} + D_{b,c}$  are equal.

An easy variant of the theorems above is the following:

**Theorem 5.** For any family of real numbers  $\{D_I\}_{I \in \binom{\{1, \dots, n\}}{2}}$ , there exists an internal-positive weighted tree  $\mathcal{T}$  with leaves  $1, \dots, n$  such that  $D_I(\mathcal{T}) = D_I$  for any  $I \in \binom{\{1, \dots, n\}}{2}$  if and only if the 4-point condition holds.

In fact, if the 4-point condition holds, in particular the relaxed 4-point condition holds, so by Theorem 4, there exists a weighted tree  $\mathcal{T}$  with leaves  $1, \dots, n$  and with 2-weights equal to the  $D_I$ ; it is easy to see that, since the 4-point condition holds, the weights of the internal edges of  $\mathcal{T}$  are nonnegative; by contracting the edges of weight 0, we get an ip-weighted tree with leaves  $1, \dots, n$  and with 2-weights equal to the  $D_I$ .

For higher  $k$  the literature is more recent, see [1], [4], [9], [10], [11], [12], [14], [15], [16]. Three of the most important results for higher  $k$  are the following:

**Theorem 6. (Herrmann, Huber, Moulton, Spillner, [9]).** *If  $n \geq 2k$ , a family of positive real numbers  $\{D_I\}_{I \in \binom{\{1, \dots, n\}}{k}}$  is ip-l-treelike if and only if its restriction to every  $2k$ -subset of  $\{1, \dots, n\}$  is ip-l-treelike.*

**Theorem 7. (Pachter-Speyer, [14]).** *Let  $k, n \in \mathbb{N}$  with  $3 \leq k \leq (n+1)/2$ . A positive-weighted tree  $\mathcal{T}$  with leaves  $1, \dots, n$  and no vertices of degree 2 is determined by the values  $D_I(\mathcal{T})$ , where  $I$  varies in  $\binom{\{1, \dots, n\}}{k}$ .*

**Theorem 8. (Levy-Yoshida-Pachter, [11])** *Let  $\mathcal{T} = (T, w)$  be a positive-weighted tree with  $L(T) = \{1, \dots, n\}$ . For any distinct  $i, j \in \{1, \dots, n\}$ , define*

$$S_{i,j} = \sum_{Y \in \binom{\{1, \dots, n\} - \{i,j\}}{k-2}} D_{i,j,Y}(\mathcal{T}).$$

*Then there exists a positive-weighted tree  $\mathcal{T}' = (T', w')$  such that  $D_{i,j}(\mathcal{T}') = S_{i,j}$  for all distinct  $i, j \in \{1, \dots, n\}$ , the quartet system of  $T'$  is contained in the quartet system of  $T$  and, defined  $T_{\leq s}$  the subforest of  $T$  whose edge set consists of edges whose removal results in one of the components having size at most  $s$ , we have  $T_{\leq n-k} \cong T'_{\leq n-k}$ .*

Moreover Levy, Yoshida and Pachter proposed a neighbor-joining algorithm for reconstructing trees from  $k$ -weights. To prove the first statement of Theorem 8, Levy, Yoshida and Pachter proved that the  $S_{i,j}$  satisfy the 4-point condition. It is natural to wonder if the 4-point condition for the  $S_{i,j}$  and some other possible conditions could be sufficient for a family  $\{D_I\}_{I \in \binom{\{1, \dots, n\}}{k}}$  to be l-treelike. An easy argument about the numbers of the  $k$ -weights, the numbers of the equations given by the 4-point condition and the numbers of edges of a tree with  $n$  leaves suggests that the 4-point condition for the  $S_{i,j}$  cannot be sufficient to characterize l-treelike families. In this paper, by using the  $S_{i,j}$  defined by Levy, Yoshida and Pachter, we give a characterization of families of real numbers parametrized by  $\binom{\{1, \dots, n\}}{k}$  that are the families of  $k$ -weights of ip-weighted trees with leaf set equal to  $\{1, \dots, n\}$  (see Theorem 20).

## 2 Notation and some remarks

**Notation 9. •** *For any  $n \in \mathbb{N}$  with  $n \geq 1$ , let  $[n] = \{1, \dots, n\}$ .*

**•** *For any set  $S$  and  $k \in \mathbb{N}$ , let  $\binom{S}{k}$  be the set of the  $k$ -subsets of  $S$ .*

- Let  $S$  be a set and  $f : S \rightarrow \mathbb{R}$  be a function. For any  $A, B$  subsets of  $S$  and any  $a, b \in \mathbb{R}$ , we denote  $a \sum_{x \in A} f(x) + b \sum_{x \in B} f(x)$  by

$$\left( a \sum_{x \in A} + b \sum_{x \in B} \right) f(x).$$

- For any set  $S$  and any  $i \in S$  and  $X \subset S$ , we write  $iX$  instead of  $\{i\} \cup X$ .
- Throughout the paper, the word “tree” will denote a finite tree.
- A **node** of a tree is a vertex of degree greater than 2.
- Let  $F$  be a leaf of a tree  $T$ . Let  $N$  be the node such that the path  $p$  between  $N$  and  $F$  does not contain any node apart from  $N$ . We say that  $p$  is the **twig** associated to  $F$ . We say that an edge is **internal** if it is not an edge of a twig. We denote by  $\mathring{E}(T)$  the set of the internal edges of  $T$ .
- We say that a tree is **essential** if it has no vertices of degree 2.
- If  $a$  and  $b$  are vertices of a tree, we denote by  $p(a, b)$  the path between  $a$  and  $b$ .
- Let  $T$  be a tree and let  $S$  be a subset of  $L(T)$ . We denote by  $T|_S$  the minimal subtree of  $T$  whose set of vertices contains  $S$ . If  $\mathcal{T} = (T, w)$  is a weighted tree, we denote by  $\mathcal{T}|_S$  the tree  $T|_S$  with the weighting induced by  $w$ .
- Let  $T$  be a tree,  $T'$  be a subtree of  $T$  and  $S$  be a subtree of  $T'$ . Let  $x \in L(T) - L(T')$ . We say that  $x$  **clings to**  $S$  as to  $T'$  if the minimal subtree of  $T$  containing  $S$  and  $x$  has no edges in common with the complementary of  $S$  in  $T'$ . See Figure 2 for an example: let  $T$  be the tree in the figure and let  $T' = T|_{a,b,c,d}$  and  $S = p(a, b)$ .

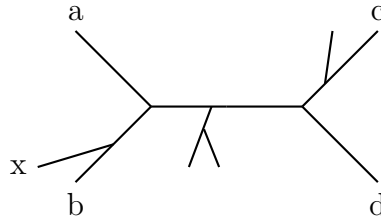


Figure 1:  $x$  clings to  $S := p(a, b)$  as to  $T' := T|_{a,b,c,d}$

**Definition 10.** Let  $T$  be a tree.

We say that two leaves  $i$  and  $j$  of  $T$  are **neighbours** if in  $p(i, j)$  there is only one node; furthermore, we say that  $C \subset L(T)$  is a **cherry** if any  $i, j \in C$  are neighbours.

The **stalk** of a cherry is the unique node in the path with endpoints any two elements of the cherry. Let  $a, b, c, d \in L(T)$ . We say that  $\langle a, b | c, d \rangle$  holds if in  $T|_{\{a,b,c,d\}}$  we have that  $a$  and  $b$  are neighbours,  $c$  and  $d$  are neighbours, and  $a$  and  $c$  are not neighbours; in this case we denote by  $\gamma_{a,b,c,d}$  the path between the stalk  $s_{a,b}$  of  $\{a, b\}$  and the stalk  $s_{c,d}$  of  $\{c, d\}$  in  $T|_{\{a,b,c,d\}}$ ; we call it the **bridge** of the quartet  $(a, b, c, d)$ . The symbol  $\langle a, b | c, d \rangle$  is called **Buneman's index** of  $a, b, c, d$ .

**Definition 11.** Let  $k \in \mathbb{N} - \{0\}$ . We say that a tree  $P$  is a **pseudostar** of kind  $(n, k)$  if  $\#L(P) = n$  and any edge of  $P$  divides  $L(P)$  into two sets such that at least one of them has cardinality greater than or equal to  $k$ .

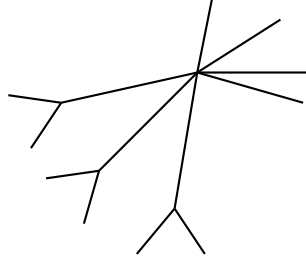


Figure 2: A pseudostar of kind  $(10, 8)$

**Remark 12.** (i) A pseudostar of kind  $(n, n-1)$  is a star, that is, a tree with only one node.  
(ii) Let  $k, n \in \mathbb{N} - \{0\}$ . If  $\frac{n}{2} \geq k$ , then every tree with  $n$  leaves is a pseudostar of kind  $(n, k)$ , in fact, if we divide a set with  $n$  elements into two parts, at least one of them has cardinality greater than or equal to  $\frac{n}{2}$ , which is greater than or equal to  $k$ .

**Theorem 13. (Baldisserrri-Rubei, [2])** Let  $n, k \in \mathbb{N}$  with  $3 \leq k \leq n-1$ . Let  $\{D_I\}_{I \in \binom{[n]}{k}}$  be a family of real numbers. If it is  $l$ -treelike, then there exists exactly one internal-nonzero-weighted essential pseudostar  $\mathcal{P}$  of kind  $(n, k)$  realizing the family.  
If the family  $\{D_I\}_{I \in \binom{[n]}{k}}$  is  $p$ - $l$ -treelike, then  $\mathcal{P}$  is positive-weighted.

### 3 Characterization of treelike families

**Definition 14.** Let  $\{D_I\}_{I \in \binom{[n]}{k}}$  be a family of real numbers. For any distinct  $i, j \in [n]$ , define

$$S_{i,j} = \sum_{Y \in \binom{[n] - \{i,j\}}{k-2}} D_{i,j,Y}$$

**Definition 15.** Let  $\{S_{i,j}\}$  for  $i, j \in [n]$  with  $i \neq j$  be a family of real numbers. For any distinct  $a, b, c, d \in [n]$ , let

$$L_{\{a,b\}}^{\{a,b,c,d\}} = \left\{ x \in [n] - \{a, b, c, d\} \mid \begin{array}{l} \text{either } S_{x,z} - S_{a,z} \text{ does not depend on } z \in \{b, c, d\} \\ \text{or } S_{x,z} - S_{b,z} \text{ does not depend on } z \in \{a, c, d\} \end{array} \right\}.$$

We will denote  $L_{\{a,b\}}^{\{a,b,c,d\}}$  simply by  $L_{a,b}^{a,b,c,d}$  and we will omit the superscript when the 4-set which we are referring to is clear from the context.

**Proposition 16.** Let  $\mathcal{A} = (A, w)$  be an internal-positive-weighted essential tree. Let  $S_{i,j}$  for distinct  $i, j \in [n]$  be the 2-weights of  $\mathcal{A}$ . Let  $a, b, c, d \in [n]$ .

- 1) If  $\langle a, b \mid c, d \rangle$  holds, we have that  $L_{a,b}$  is the set of the elements  $x$  of  $[n]$  clinging to  $p(a, b)$  as to  $T|_{a,b,c,d}$  and  $L_{c,d}$  is the set of the elements  $x$  of  $[n]$  clinging to  $p(c, d)$  as to  $T|_{a,b,c,d}$ .
- 2) We have that  $\langle a, b \mid c, d \rangle$  holds and the bridge of  $(a, b, c, d)$  is given by exactly one edge if and only if the following conditions hold:

(i)

$$S_{a,b} + S_{c,d} < S_{a,c} + S_{b,d} = S_{a,d} + S_{b,c}$$

(ii)  $L_{a,b} \cup L_{c,d} = [n]$ .

*Proof.* 1) Observe that  $L_{a,b}$  is the set of the elements  $x$  of  $[n]$  that are neighbours either of  $a$  or of  $b$  in  $A|_{a,b,c,d,x}$ ; hence, if  $\langle a, b | c, d \rangle$  holds, we have that  $L_{a,b}$  is the set of the elements  $x$  of  $[n]$  clinging to  $p(a, b)$  as to  $A|_{a,b,c,d}$ .

2)  $\implies$  Suppose  $\langle a, b | c, d \rangle$  holds and the bridge of  $(a, b, c, d)$  is given by exactly one edge; then the weight of the bridge is positive, so (i) holds; moreover,  $L_{a,b}$  is the set of the elements  $x$  of  $[n]$  clinging to  $p(a, b)$  and  $L_{c,d}$  is the set of the elements  $x$  of  $[n]$  clinging to  $p(c, d)$ . So (ii) must hold.

$\Leftarrow$  If  $\langle a, b | c, d \rangle$  did not hold, then either  $A|_{a,b,c,d}$  would be a star or one of  $\langle a, c | b, d \rangle$  and  $\langle a, d | b, c \rangle$  would hold. So we would have either  $S_{a,b} + S_{c,d} = S_{a,d} + S_{b,c}$  or  $S_{a,b} + S_{c,d} = S_{a,c} + S_{b,d}$ , which is absurd by assumption (i). Hence  $\langle a, b | c, d \rangle$  holds. Moreover, if the bridge of  $(a, b, c, d)$  were given by more than one edge, then, since  $A$  is essential, there would exist  $x \in [n]$  clinging to the bridge, and so we would have  $x \notin L_{a,b} \cup L_{c,d}$ , which is absurd by condition (ii).  $\square$

**Remark 17.** Let  $\mathcal{T} = (T, w)$  be a weighted essential tree with  $L(T) = [n]$  and let  $k$  be a natural number less than  $n$ . Let  $e_i$  denote the twig associated to  $i$  for any  $i \in [n]$ . Then

$$w(e_i) = \frac{D_I(\mathcal{T})}{k} - \frac{1}{k} \sum_{e \in \dot{E}(T|_I)} w(e) + \frac{1}{k} \sum_{j \in I} \left( D_{j,X}(\mathcal{T}) - D_{i,X}(\mathcal{T}) - \sum_{e \in \dot{E}(T|_{jX})} w(e) + \sum_{e \in \dot{E}(T|_{iX})} w(e) \right)$$

for any  $i \in [n]$ ,  $I \in \binom{[n]}{k}$  and  $X = X_{i,j} \in \binom{[n] - \{i,j\}}{k-1}$  (depending on  $i$  and  $j$ ).

*Proof.* Let  $I \in \binom{[n]}{k}$ . Then

$$D_I(\mathcal{T}) = \sum_{i \in I} w(e_i) + \sum_{e \in \dot{E}(T|_I)} w(e). \quad (1)$$

Thus, for any  $i, j \in [n]$ ,

$$w(e_j) - w(e_i) = D_{j,X}(\mathcal{T}) - D_{i,X}(\mathcal{T}) - \sum_{e \in \dot{E}(T|_{jX})} w(e) + \sum_{e \in \dot{E}(T|_{iX})} w(e) \quad (2)$$

for any  $X \in \binom{[n]}{k-1}$  such that  $i, j \notin X$ . Obviously, for any  $i \in [n]$  and any  $I \in \binom{[n]}{k}$ , we have:

$$k w(e_i) = \sum_{j \in I} (w(e_i) - w(e_j)) + \sum_{j \in I} w(e_j). \quad (3)$$

From (1), (2) and (3), we get easily our assertion.  $\square$

**Proposition 18.** Let  $\mathcal{T}$  be an  $ip$ -weighted tree with  $L(T) = [n]$ . Let  $S_{i,j}$  for distinct  $i, j \in [n]$  be defined from the  $D_I(\mathcal{T})$  for  $I \in \binom{[n]}{k}$  as in Definition 14. Let  $\mathcal{T}'$  be an  $ip$ -weighted tree with 2-weights the  $S_{i,j}$  (the existence follows from Theorem 5 and paper [11], in particular Corollary 11, where the assumption that all the weights of  $\mathcal{T}$  are positive is not necessary). Then  $\mathcal{T}'$  is a pseudostar of kind  $(n, k)$ .

*Proof.* Suppose, contrary to our claim, that there exists an edge  $e$  of  $T'$  dividing  $L(T') = [n]$  into two parts both of cardinality less than  $k$ . By Theorem 8, or more precisely by the analogous statement for ip-weighted trees, the quartet system of  $T'$  is contained in the quartet system of  $T$ , so  $T'$  is obtained from  $T$  by contracting some edges (see Theorem 1 in [5]); thus  $e$  corresponds to an edge of  $T$  dividing  $L(T) = [n]$  into two parts both of cardinality less than  $k$ . We can suppose  $e$  is  $\gamma_{a,b,c,d}$  for some  $a, b, c, d \in [n]$  such that  $\langle a, b | c, d \rangle$  holds. Denote  $s_{a,b}$  by  $t$  and  $s_{c,d}$  by  $s$  (see Definition 10). We want to show that

$$S_{a,b} + S_{c,d} = S_{a,c} + S_{b,d} \quad (4)$$

(which is absurd since it implies that the weight of  $e$  is equal to 0). Obviously  $S_{a,b}$  is equal to

$$\sum_{E \in \binom{[n] - \{a,b,c,d\}}{k-2}} D_{a,b,E}(\mathcal{T}) + \sum_{E \in \binom{[n] - \{a,b,c,d\}}{k-3}} D_{a,b,c,E}(\mathcal{T}) + \sum_{E \in \binom{[n] - \{a,b,c,d\}}{k-3}} D_{a,b,d,E}(\mathcal{T}) + \sum_{E \in \binom{[n] - \{a,b,c,d\}}{k-4}} D_{a,b,c,d,E}(\mathcal{T})$$

and analogously  $S_{c,d}$ ,  $S_{a,c}$  and  $S_{b,d}$ . Hence (4) is equivalent to

$$\sum_{E \in \binom{[n] - \{a,b,c,d\}}{k-2}} (D_{a,b,E}(\mathcal{T}) + D_{c,d,E}(\mathcal{T})) = \sum_{E \in \binom{[n] - \{a,b,c,d\}}{k-2}} (D_{a,c,E}(\mathcal{T}) + D_{b,d,E}(\mathcal{T})). \quad (5)$$

We can write  $E \in \binom{[n] - \{a,b,c,d\}}{k-2}$  as disjoint union of  $E_a, E_b, E_c, E_d, E_t, E_s$ , where:

$E_a = \{x \in E \mid x \text{ clings to } p(a, t) - \{t\} \text{ as to } T|_{a,b,c,d}\}$

and analogously  $E_b, E_c, E_d$ ,

$E_t = \{x \in E \mid x \text{ clings to } t \text{ as to } T|_{a,b,c,d}\}$

and analogously  $E_s$ . By our assumption that  $e$  divides  $L(T) = [n]$  into two parts both of cardinality less than  $k$ , we have that  $E_a \cup E_b \cup E_t \neq \emptyset$  and  $E_c \cup E_d \cup E_s \neq \emptyset$ , in fact: define  $A = E_a \cup E_b \cup E_t \cup \{a, b\}$  and  $B = E_c \cup E_d \cup E_s \cup \{c, d\}$ ; we have that  $E \cup \{a, b, c, d\}$  is the (disjoint) union of  $A$  and  $B$ , hence  $\#(A \cup B) = \#(E \cup \{a, b, c, d\}) = k + 2$ ; moreover  $\#A \leq k - 1$ ,  $\#B \leq k - 1$ , therefore  $\#A \geq 3$  and  $\#B \geq 3$ , which gives the desired conclusion.

So we get:

$$\begin{aligned} D_{a,b,E}(\mathcal{T}) &= w(p(a, t)) + w(p(b, t)) + w(e) + w(p(s, \overline{E_c})) + w(p(s, \overline{E_d})) \\ &\quad + w(T|_{E_a, t} - p(a, b)) + w(T|_{E_b, t} - p(a, b)) + w(T|_{E_c, s} - p(c, d)) + w(T|_{E_d, s} - p(c, d)), \end{aligned}$$

where  $\overline{E_c}$  is the vertex of  $T|_{E_c, s} \cap p(s, c)$  which is the most far from  $s$  and analogously  $\overline{E_d}$ . We can write  $D_{a,c,E}(\mathcal{T})$ ,  $D_{b,d,E}(\mathcal{T})$ ,  $D_{c,d,E}(\mathcal{T})$  in an analogous way and we get that

$$D_{a,b,E}(\mathcal{T}) + D_{c,d,E}(\mathcal{T}) = D_{a,c,E}(\mathcal{T}) + D_{b,d,E}(\mathcal{T}).$$

So (5) holds. □

**Remark 19.** Let  $\mathcal{T}$  be an ip-weighted tree with  $L(T) = [n]$ . Let  $S_{i,j}$  for  $i, j \in [n]$  be defined from the  $D_I(\mathcal{T})$  for  $I \in \binom{[n]}{k}$  as in Definition 14. Let  $\mathcal{T}' = (T', w')$  be an essential ip-weighted tree with  $\{S_{i,j}\}$  as family of the 2-weights (the existence follows from Theorem 5 and paper [11], in particular Corollary 11). It is a pseudostar of kind  $(n, k)$  by Proposition 18 (so it is equal to  $T'_{\leq n-k} \cong T_{\leq n-k}$ ).

Let  $e$  be an internal edge of  $T'$  and let  $a, b, c, d \in [n]$  such that  $\langle a, b \mid c, d \rangle$  holds and the bridge of  $(a, b, c, d)$  is given only by the edge  $e$ ; in [11] (in particular see Lemma 12), the authors proved that

$$w(e) = \frac{2w'(e)}{\binom{\#L_{a,b}-2}{k-2} + \binom{\#L_{c,d}-2}{k-2}}$$

**Theorem 20.** Let  $\{D_I\}_{I \in \binom{[n]}{k}}$  be a family in  $\mathbb{R}$ . Let  $S_{i,j}$  be defined from the family  $\{D_I\}_{I \in \binom{[n]}{k}}$  as in Definition 14 and see Definition 15 for the definition of  $L_{a,b}$ . For any  $W \in \binom{[n]}{k}$ , let us denote

$$Q(W) = \left\{ (a, b, c, d) \in \binom{W}{4} \mid S_{a,b} + S_{c,d} < S_{a,c} + S_{b,d} = S_{a,d} + S_{b,c}, \quad L_{a,b} \cup L_{c,d} = [n] \right\} / \sim$$

where  $(a, b, c, d) \sim (a', b', c', d')$  if and only if, up to swapping  $\{a, b\}$  with  $\{c, d\}$ , we have that

$$\{a, b\} \subset L_{a',b'}, \quad \{c, d\} \subset L_{c',d'}, \quad \{a', b'\} \subset L_{a,b}, \quad \{c', d'\} \subset L_{c,d}$$

The family  $\{D_I\}_{I \in \binom{[n]}{k}}$  is *ip-l-treelike* if and only if the following conditions hold:

- (i) the  $S_{i,j}$  for distinct  $i, j \in [n]$  satisfy the 4-point condition;
- (ii) for any distinct  $a, b, c, d \in [n]$  such that  $S_{a,c} + S_{b,d} = S_{a,d} + S_{b,c}$ ,  $L_{a,b} \cup L_{c,d} = [n]$  and  $\#L_{a,b} < k$ ,  $\#L_{c,d} < k$ , we have:

$$S_{a,b} + S_{c,d} = S_{a,c} + S_{b,d}; \tag{6}$$

- (iii) for any  $I \in \binom{[n]}{k}$ , we have:

$$\sum_{i,j \in I} \left[ \frac{D_{j,X} - D_{i,X}}{2} + \left( \sum_{[(a,b,c,d)] \in Q(iX)} - \sum_{[(a,b,c,d)] \in Q(jX)} \right) \frac{S_{a,c} + S_{b,d} - S_{a,b} - S_{c,d}}{\binom{\#L_{a,b}-2}{k-2} + \binom{\#L_{c,d}-2}{k-2}} \right] = 0,$$

where  $X = X_{i,j}$  is any element of  $\binom{[n]-\{i,j\}}{k-1}$  (so it depends on  $i$  and  $j$ ).

- (iv) for any  $i \in [n]$ ,

$$D_I + \sum_{j \in I} (D_{j,X} - D_{i,X}) + \left[ \sum_{j \in I} \left( \sum_{[(a,b,c,d)] \in Q(iX)} - \sum_{[(a,b,c,d)] \in Q(jX)} \right) - \sum_{[(a,b,c,d)] \in Q(I)} \right] \frac{S_{a,c} + S_{b,d} - S_{a,b} - S_{c,d}}{\binom{\#L_{a,b}-2}{k-2} + \binom{\#L_{c,d}-2}{k-2}}$$

does not depend on  $I \in \binom{[n]}{k}$  and  $X = X_{i,j} \in \binom{[n]-\{i,j\}}{k-1}$ .

*Proof.*  $\implies$  Let  $\mathcal{T} = (T, w)$  be an ip-weighted tree with  $L(T) = [n]$  and realizing the family. Condition (i) has been proved in [11]. Condition (ii) follows from Propositions 16 and 18 and (i), in fact: let  $a, b, c, d$  be distinct elements of  $[n]$  such that  $S_{a,c} + S_{b,d} = S_{a,d} + S_{b,c}$ ,  $L_{a,b} \cup L_{c,d} = [n]$ ,  $\#L_{a,b} < k$  and  $\#L_{c,d} < k$ ; by condition (i), we have that  $S_{a,b} + S_{c,d} \leq S_{a,c} + S_{b,d}$ ; let  $\mathcal{T}' = (T', w')$  be the ip-weighted essential tree with  $L(T') = [n]$  and such that the 2-weights are equal to the  $S_{i,j}$ ; if, contrary to our claim, we had  $S_{a,b} + S_{c,d} < S_{a,c} + S_{b,d}$ , then, by Proposition 16,  $\langle a, b \mid c, d \rangle$  would hold and the bridge of  $(a, b, c, d)$  would be given by exactly one edge; since  $\#L_{a,b} < k$  and  $\#L_{c,d} < k$ , we would have that  $\mathcal{T}'$  is not a pseudostar of kind  $(n, k)$ , but this is absurd by Proposition 18.



Let us prove (iii). We have:

$$\begin{aligned}
D_I(\mathcal{T}) &= \sum_{i \in I} w(e_i) + \sum_{e \in \mathring{E}(T|_I)} w(e) = \\
&= \sum_{i \in I} \left( \frac{D_I(\mathcal{T})}{k} - \frac{1}{k} \sum_{e \in \mathring{E}(T|_I)} w(e) + \frac{1}{k} \sum_{j \in I} \left( D_{j,X}(\mathcal{T}) - D_{i,X}(\mathcal{T}) - \sum_{e \in \mathring{E}(T|_{jX})} w(e) + \sum_{e \in \mathring{E}(T|_{iX})} w(e) \right) \right) + \\
&\quad + \sum_{e \in \mathring{E}(T|_I)} w(e) = \\
&= D_I(\mathcal{T}) - \sum_{e \in \mathring{E}(T|_I)} w(e) + \frac{1}{k} \sum_{i,j \in I} \left( D_{j,X}(\mathcal{T}) - D_{i,X}(\mathcal{T}) - \sum_{e \in \mathring{E}(T|_{jX})} w(e) + \sum_{e \in \mathring{E}(T|_{iX})} w(e) \right) + \sum_{e \in \mathring{E}(T|_I)} w(e),
\end{aligned}$$

where  $X = X_{i,j}$  is any element of  $\binom{[n] - \{i,j\}}{k-2}$  (so it depends on  $i$  and  $j$ ) and the second equality holds by Remark 17. Hence

$$\sum_{i,j \in I} \left( D_{j,X}(\mathcal{T}) - D_{i,X}(\mathcal{T}) - \sum_{e \in \mathring{E}(T|_{jX})} w(e) + \sum_{e \in \mathring{E}(T|_{iX})} w(e) \right) = 0. \quad (7)$$

By Proposition 16, we have that, for any distinct  $a, b, c, d \in [n]$ , we have that  $\langle a, b | c, d \rangle$  holds and the bridge of  $(a, b, c, d)$  is given by exactly one edge if and only if  $S_{a,b} + S_{c,d} < S_{a,c} + S_{b,d} = S_{a,d} + S_{b,c}$  and  $L_{a,b} \cup L_{c,d} = [n]$ . Moreover, given  $a, b, c, d, a', b', c', d'$  such that  $\langle a, b | c, d \rangle$  holds and the bridge of  $(a, b, c, d)$  is given by exactly one edge and analogously for  $a', b', c', d'$ , we have that  $(a, b, c, d)$  and  $(a', b', c', d')$  give the same edge if and only if they are equivalent. So, for any  $W \in \binom{[n]}{k}$ , an internal edge of  $\mathcal{T}|_W$  corresponds to an element of  $Q(W)$ . Hence, from (7) and Lemma 12 in [11] (see Remark 19), we get condition (iii). Finally, by Remark 17, the fact that an internal edge of  $\mathcal{T}|_W$  corresponds to an element of  $Q(W)$  and Lemma 12 in [11] (see Remark 19), we get (iv).

$\Leftarrow$  Let  $\mathcal{T}' = (T', w')$  be an essential ip-weighted tree with 2-weights equal to the  $S_{i,j}$  (it exists by condition (i) and Theorem 5). It is a pseudostar of kind  $(n, k)$  by condition (ii), in fact: let  $e$  be an internal edge of  $T'$ ; let  $a, b, c, d \in [n]$  be such that  $\langle a, b | c, d \rangle$  holds and the bridge of  $(a, b, c, d)$  is given only by  $e$ ; then  $S_{a,b} + S_{c,d} < S_{a,c} + S_{b,d} = S_{a,d} + S_{b,c}$  and  $L_{a,b} \cup L_{c,d} = [n]$ ; if, contrary to our claim, we had  $\#L_{a,b} < k$ ,  $\#L_{c,d} < k$ , then by (ii), we would get a contradiction.

Let  $\mathcal{T} = (T, w)$  be the weighted tree with  $T = T'$  and where  $w$  is defined as follows: for any  $e \in \mathring{E}(T')$ , let  $a, b, c, d \in [n]$  be such that  $\langle a, b | c, d \rangle$  holds and the bridge of  $(a, b, c, d)$  is  $e$ ; define

$$w(e) = \frac{2w'(e)}{\binom{\#L_{a,b}-2}{k-2} + \binom{\#L_{c,d}-2}{k-2}};$$

hence

$$w(e) = \frac{S_{a,c} + S_{b,d} - S_{a,b} - S_{c,d}}{\binom{\#L_{a,b}-2}{k-2} + \binom{\#L_{c,d}-2}{k-2}};$$

moreover, for any  $i \in [n]$ , define

$$w(e_i) = \frac{D_I}{k} - \frac{1}{k} \sum_{e \in \dot{E}(T|_I)} w(e) + \frac{1}{k} \sum_{j \in I} \left( D_{j,X} - D_{i,X} - \sum_{e \in \dot{E}(T|_{jX})} w(e) + \sum_{e \in \dot{E}(T|_{iX})} w(e) \right)$$

for any  $I \in \binom{[n]}{k}$  and any  $X = X_{i,j} \in \binom{[n] - \{i,j\}}{k-1}$  (so it depends on  $i$  and  $j$ ). Observe that it is a good definition by condition (iv). We have to show that  $D_I(\mathcal{T}) = D_I$  for any  $I \in \binom{[n]}{k}$ . We have:

$$\begin{aligned} D_I(\mathcal{T}) &= \sum_{i \in I} w(e_i) + \sum_{e \in \dot{E}(T|_I)} w(e) = \\ &= \sum_{i \in I} \left( \frac{D_I}{k} - \frac{1}{k} \sum_{e \in \dot{E}(T|_I)} w(e) + \frac{1}{k} \sum_{j \in I} \left( D_{j,X} - D_{i,X} - \sum_{e \in \dot{E}(T|_{jX})} w(e) + \sum_{e \in \dot{E}(T|_{iX})} w(e) \right) \right) + \\ &\quad + \sum_{e \in \dot{E}(T|_I)} w(e) = \\ &= D_I - \sum_{e \in \dot{E}(T|_I)} w(e) + \frac{1}{k} \sum_{i,j \in I} \left( D_{j,X} - D_{i,X} - \sum_{e \in \dot{E}(T|_{jX})} w(e) + \sum_{e \in \dot{E}(T|_{iX})} w(e) \right) + \sum_{e \in \dot{E}(T|_I)} w(e), \end{aligned}$$

where  $X = X_{i,j}$  is any element of  $\binom{[n] - \{i,j\}}{k-2}$  (so it depends on  $i$  and  $j$ ) and the second equality holds by the definition of  $w(e_i)$ . So  $D_I(\mathcal{T}) = D_I$  if and only if

$$\sum_{i,j \in I} \left( D_{j,X} - D_{i,X} - \sum_{e \in \dot{E}(T|_{jX})} w(e) + \sum_{e \in \dot{E}(T|_{iX})} w(e) \right) = 0,$$

which is true by the definition of the weight of the internal edges and by assumption (iii).  $\square$

**Remark 21.** *It is easy to get from Theorem 20 a characterization also for  $p$ -l-treelike families. Obviously a family  $\{D_I\}_{I \in \binom{[n]}{k}}$  is  $p$ -l-treelike if and only if conditions (i), (ii), (iii), (iv) of Theorem 20 hold and, in addition, the number displayed in (iv) is positive for any  $i \in [n]$ .*

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